I. THE NEWTONIAN LIMIT OF GENERAL RELATIVITY (8 POINTS)

Write the Newtonian limit line element as

$$ds^2 = (-\varepsilon^{-2} - 2\Phi)dt^2 + \delta_{ij}dx^i dx^j + O(\varepsilon^2),$$  \hspace{1cm} (1.1)

where $\Phi = \Phi(x^\alpha)$, is the usual Newtonian potential. When the metric is diagonal, there is a useful shortcut to the geodesic equations and the connection coefficients. Recall from Eq. (2.4) on the solutions to homework 4, that we can write the geodesic equations as

$$g^{\gamma\lambda} \ddot{x}_\lambda = \frac{1}{2} g_{\alpha\beta,\gamma} - g_{\gamma\alpha,\beta} \dot{x}_\alpha \dot{x}_\beta,$$  \hspace{1cm} (1.2)

where here an overdot represents $d/d\tau$. We can now quickly derive a second order equation for each coordinate by choosing different values for $\gamma$. Putting $\gamma = t$, we find

$$\ddot{t} = 0 + O(\varepsilon^2).$$  \hspace{1cm} (1.3)

But since the geodesic equations can be written as $\ddot{x}^\alpha = -\Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma$, this means

$$\Gamma^t_{\alpha\beta} = 0 + O(\varepsilon^2).$$  \hspace{1cm} (1.4)

Similarly, putting $\gamma = i$, in Eq. (1.2) gives

$$\ddot{x}^i = \frac{1}{2} g_{tt,i} \dot{t}^2 = -\Phi_{,i} \dot{t}^2 + O(\varepsilon^2),$$  \hspace{1cm} (1.5)

since both $g_{\alpha\beta}$ and $g_{jk,i}$ vanish at this order. Again we can easily extract the connection coefficients by comparing to the geodesic equation

$$\Gamma^i_{tt} = \Phi_{,i} + O(\varepsilon^2), \quad \Gamma^i_{jk} = 0 + O(\varepsilon^2).$$  \hspace{1cm} (1.6)

By combining Eqs. (1.4) and (1.6) we see that the connection coefficients have the form

$$\Gamma^\alpha_{\beta\gamma} = (0)\Gamma^\alpha_{\beta\gamma} + O(\varepsilon^2)$$  \hspace{1cm} (1.7)

where the only non-vanishing $(0)\Gamma^\alpha_{\beta\gamma}$ are

$$(0)\Gamma^i_{tt} = \Phi_{,i}. \hspace{1cm} (1.8)$$

Note that Eq. (1.3) gives $\ddot{t} = 0 + O(\varepsilon^2)$, which means that $\dot{t} = \gamma + O(\varepsilon^2)$, where $\gamma$ is a constant. Then we can write proper time derivatives in terms of coordinate time and the constant $\gamma$. For example,

$$\ddot{x}^i = \frac{d}{d\tau} \left( \frac{dx^i}{d\tau} \right) = \frac{dt}{d\tau} \frac{d}{dt} \left( \frac{dt}{d\tau} \frac{dx^i}{dt} \right) = \gamma^2 \frac{d^2x^i}{dt^2} + O(\varepsilon^2).$$  \hspace{1cm} (1.9)

Using this in the geodesic equation (1.5), along with $\ddot{t}^2 = \gamma^2$, we recover the familiar Newtonian relation

$$\frac{d^2x}{dt^2} = -\nabla \Phi + O(\varepsilon^2).$$  \hspace{1cm} (1.10)
II. HARTLE CHAPTER 9, PROBLEM 15 (10 POINTS)

As suggested in Hartle, we will explicitly show factors of $G$ and $c$ (i.e. use units where $G \neq 1$ and $c \neq 1$) in this problem. The goal of this problem is to calculate the leading order correction (in $c^{-4}$ as written here) to the change in $\phi$ during one complete cycle of the radial motion

\[
\Delta \phi = \int_0^{T_r} d\tau \frac{d\phi}{d\tau} = 2 \int_{r_1}^{r_2} dr \left( \frac{dr}{d\tau} \right)^{-1} \frac{d\phi}{d\tau} = 2l \int_{r_1}^{r_2} \frac{dr}{r^2} \left( 1 - \frac{2GM}{c^2r} \right)^{-1/2} \left[ c^2c^2 \left( 1 - \frac{2GM}{c^2r} \right)^{-1} - c^2 - \frac{l^2}{r^2} \right]^{-1/2}.
\]

(2.1)

Here $T_r$ is the proper-period of the radial motion, and $r_{1,2}$ are turning points where $dr/d\tau$ (or equivalently the quantity in square brackets) vanishes, and we have let $r(\tau = 0) = r_1$.

A.

Recall that the energy parameter $c$ can be written in terms of a the Newtonian energy $E$

\[
e^2 = \left( 1 + \frac{E}{mc^2} \right)^2 = 1 + \frac{2E}{mc^2} + \frac{E^2}{m^2c^4}.
\]

(2.2)

We will also expand

\[
\left( 1 - \frac{2GM}{c^2r} \right)^{-1/2} = 1 + \frac{GM}{c^2r} + O(c^{-4}),
\]

(2.3)

\[
\left( 1 - \frac{2GM}{c^2r} \right)^{-1} = 1 + \frac{2GM}{c^2r} + \frac{4G^2M^2}{c^4r^2} + O(c^{-6}),
\]

(2.4)

in first two terms in the integral (2.1). We keep terms of order $c^{-4}$ in the last quantity, and $e^2$, because they are multiplied by $c^2$ in the integrand

\[
c^2c^2 \left( 1 - \frac{2GM}{c^2r} \right)^{-1} = e^2 + \frac{2GM}{c^2r} + \frac{4G^2M^2}{c^4r^2} + \frac{2E}{mc^2} + \frac{4EGM}{mc^3r} + \frac{E^2}{m^2c^4} + O(c^{-4}).
\]

(2.5)

Putting these expansions into the integral (2.1) gives

\[
\Delta \phi = 2l \int_{r_1}^{r_2} \frac{dr}{r^2} \left[ 1 + \frac{GM}{c^2r} + O(c^{-4}) \right] \left[ -A \frac{1}{r^2} + B \frac{1}{r} + C + O(c^{-4}) \right]^{-1/2},
\]

(2.6)

where the coefficients $A$, $B$, and $C$ group quantities by power of $r$,

\[
A = l^2 - 4 \left( \frac{GM}{c^2} \right)^2, \quad B = 2GM + \frac{4EGM}{mc^2}, \quad C = \frac{2E}{mc^2} + \frac{E^2}{m^2c^4}.
\]

(2.7)

B.

If we change variables to $u = 1/r$, which means $du = -dr/r^2$, we find

\[
\Delta \phi = 2l \int_{u_2}^{u_1} du \left[ 1 + \frac{GM}{c^2} u + O(c^{-4}) \right] \left[ -Au^2 + Bu + C + O(c^{-4}) \right]^{-1/2},
\]

(2.8)

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1 This is of course not necessary as the results cannot depend on our choice of units. When possible, it is best to introduce a formal expansion parameter (i.e. $\varepsilon$ in I) rather than to try to monitor more physical quantities, which are exceptionally skilled at misbehaving (see III. C. in the solutions to homework 4).
where \( u_{1,2} = 1/r_{1,2} \). If we factor \( A \) out of the denominator and expand it in the numerator, using (2.7), we find

\[
\Delta \phi = 2 \int_{u_2}^{u_1} du \left[ 1 + 2 \left( \frac{GM}{lc} \right)^2 + \frac{GM}{c^2} u + O(c^{-4}) \right] \left[ -u^2 + \frac{B}{A} u + \frac{C}{A} + O(c^{-4}) \right]^{-1/2}.
\] (2.9)

At the turning points \( u_{1,2} \), the denominator must vanish. We can rewrite the integral to emphasize this. Keeping in mind that \( u_2 < u < u_1 \)

\[
\Delta \phi = 2 \int_{u_2}^{u_1} du \left[ 1 + 2 \left( \frac{GM}{lc} \right)^2 + \frac{GM}{c^2} u + O(c^{-4}) \right] \left[ (u_1 - u)(u - u_2) + O(c^{-4}) \right]^{-1/2},
\] (2.10)

where, by comparison with Eq. (2.9)

\[
\frac{u_1 + u_2}{A} = 2 \frac{GM}{l^2} + O(c^{-2}), \quad u_1 u_2 = - \frac{C}{A} = - \frac{2E}{ml^2} + O(c^{-2}).
\] (2.11)

This leaves us with the expected result

\[
\Delta \phi = \left[ 1 + 2 \left( \frac{GM}{lc} \right)^2 \right] 2 \int_{u_2}^{u_1} du \left[ \frac{du}{(u_1 - u)(u - u_2)} \right]^{1/2} + \frac{2GM}{c^2} \int_{u_2}^{u_1} du \left[ \frac{du}{(u_1 - u)(u - u_2)} \right]^{1/2} + O(c^{-4}).
\] (2.12)

C.

As mentioned in Hartle, the first integral is

\[
2 \int_{u_2}^{u_1} du \left[ \frac{du}{(u_1 - u)(u - u_2)} \right]^{1/2} = 2\pi.
\] (2.13)

The second integral in Eq. (2.12) can be symmetrized by changing variables to the distance from center of the integration region \( \bar{u} = (u_1 + u_2)/2 \)

\[
q = u - \bar{u}, \quad q(u_1) = Q = \frac{1}{2} \Delta u, \quad q(u_2) = -Q,
\] (2.14)

where \( \Delta u = u_1 - u_2 \). Then we have

\[
\int_{u_2}^{u_1} du \left[ \frac{du}{(u_1 - u)(u - u_2)} \right]^{1/2} = \int_{-Q}^{Q} dq \left[ \frac{q + \bar{u}}{(Q - q)(Q + q)} \right]^{1/2}
\] (2.15)

The first term of the integrand is odd and does not contribute since the integration bounds are symmetric about the origin. The second term, which is even, can be recognized as a rescaling of the integral which we already knew (2.13)

\[
\int_{u_2}^{u_1} du \left[ \frac{du}{(u_1 - u)(u - u_2)} \right]^{1/2} = \bar{u} (2\pi) = \frac{\pi}{2} (u_1 + u_2) \equiv \frac{\pi GM}{l^2} + O(c^{-2}),
\] (2.16)

where we have used the leading order expression for \( u_1 + u_2 \) as it appears in in (2.11).

D.

By substituting the integrals (2.13) and (2.16) into Eq. (2.12), we find that the the perihelion shift after one cycle of radial motion, \( \delta \phi = \Delta \phi - 2\pi \), is given by the expected result

\[
\delta \phi = 6\pi \left( \frac{GM}{lc} \right)^2 + O(c^{-4}).
\] (2.17)
At very large radii \( r \gg M \), Newtonian gravity is valid. The magnitude of the force due to gravity, on an observer of mass \( m \), is then given by \( F = Mm/r^2 \). If the observer has height \( h \), and is oriented feet-first, then the the force at the observer’s head is given by

\[
F_{\text{head}} \sim \frac{Mm}{(r+h)^2} \sim \frac{2Mm}{r^3} + O \left( \frac{h^2}{r^2} \right),
\]

where \( F_{\text{feet}} \sim Mm/r^2 \), is the force at the observer’s feet. Up to corrections of order \( h^2/r^2 \), the difference in forces \( \Delta = F_{\text{feet}} - F_{\text{head}} \) must always be smaller than

\[
\Delta_{\text{max}} = \frac{2Mm}{R^3} h,
\]

where \( R \) is the outer radius of spherically symmetric source of the gravity. If we want the source to be such that \( \Delta < 20M_m/m/R^2_\oplus \), which is the same as \( \Delta < 20mg \), then we must have \( 2Mm/R^3 < 20M_m/mR^2_\oplus \), or rather

\[
\frac{M}{R^3} \lesssim 10^9 \text{ g/cm}^3,
\]

where we have used \( h \sim 10^2 \text{ cm}, M_\oplus \sim 10^{28} \text{ g}, \) and \( R_\oplus \sim 10^9 \text{ cm} \). For perhaps a better perspective, we can rewrite this constraint as either of following:

\[
\frac{M}{M_\oplus} \lesssim 10^8 \left( \frac{R}{R_\oplus} \right)^3, \quad \frac{R}{R_\oplus} \gtrsim 10^{-3} \left( \frac{M}{M_\oplus} \right)^{1/3}.
\]

The largest density allowed by the \( g \)-force constraint (3.3) is considerably more than the density of the Earth, \( M_\oplus/R^3_\oplus \sim 10 \text{ g/cm}^3 \), although it is well below the typical density of a neutron star \( M_{\NS}/R^3_{\NS} \sim 10^{15} \text{ g/cm}^3 \) (where we have used \( M_{\NS}/M_\oplus \) and \( R_{\NS} \sim 10 \text{ km} \)). In fact, it is roughly the largest possible density of a white dwarf \( \max(M_{\WD}/R^3_{\WD}) \sim 10^9 \text{ g/cm}^3 \) (see figure 24.5 in Hartle).

Suppose we also enforced the “time machine” requirement of box 9.1 in Hartle: \( (1 - 2M/R)^{1/2} \sim 10^{-5} \), or rather \( R \sim 2M \). Recall that in order to use Newtonian arguments we had assumed \( r \gg M \) when deriving the \( g \)-force constraint (3.3). This assumption now reads \( r \gg R/2 \), which means that the largest \( g \)-forces (which occur at \( r \sim R \)) are not well described by the Newtonian result Eq. (3.2). The constraint (3.3) would have to be replaced by some relativistic result.

The period \( T_\infty \) of a circular orbit at constant Schwarzschild radius \( r = 7M \), as recorded by an observer at \( r \to \infty \) is given by Eq. (9.44) in Hartle

\[
T_\infty = \frac{2\pi}{\Omega} = 2\pi \left( \frac{d\phi}{dt} \right)^{-1} = 2\pi T^{3/2}M = 116M. \tag{4.1}
\]

An observer riding along this orbit however would measure a period \( T_{\text{proper}} \) given by

\[
T_{\text{proper}} = 2\pi \left( \frac{d\phi}{d\tau} \right)^{-1} = 2\pi \left( \frac{d\phi}{dt} \frac{dt}{d\tau} \right)^{-1} = T_\infty \left( \frac{dt}{d\tau} \right)^{-1} = T_\infty \left( 1 - \frac{3M}{r} \right)^{1/2} = T_\infty \sqrt{\frac{4}{7}} = 88M, \tag{4.2}
\]

since for circular orbits \( dt/d\tau = (1 - 3M/r)^{-1/2} \) [see Eq. (9.48) in Hartle]. This result agrees with the general notion that an increase in the gravitational field strength corresponds with a decrease in the rate at which clocks tick. To convert these results into more familiar units of time just replace \( M \) with \( GM/c^3 \), or equivalently multiply by 4.92 \( \mu s/M_\odot \) to find

\[
T_\infty = 571 \left( \frac{M}{M_\odot} \right) \mu s, \quad T_{\text{proper}} = 433 \left( \frac{M}{M_\odot} \right) \mu s. \tag{4.3}
\]
V. PERIHELION PRECESSION IN THE PARAMETRIZED-POST-NEWTONIAN (PPN) FORMALISM
(10 POINTS)

We will again compute the change in \( \phi \) during one complete cycle of the radial motion

\[
\Delta \phi = \int_0^{T_r} \frac{d\phi}{d\tau} \, d\tau = 2 \int_{r_1}^{r_2} \frac{dr}{dr} \left( \frac{dr}{d\tau} \right)^{-1} \frac{d\phi}{d\tau},
\]

(5.1)

to order \( c^{-2} \). As before \( T_r \) is the proper-period of the radial motion, \( r_{1,2} \) are turning points where \( dr/d\tau = 0 \), and we let \( r(\tau = 0) = r_1 \). Unlike before however, we assume a general PPN spacetime of the form (in the “equatorial plane” \( \theta = \pi/2 \))

\[
ds^2 = -A(r)c^2 \, dt^2 + B(r) \, dr^2 + r^2 \, d\theta^2 + r^2 \, d\phi^2, \tag{5.2}
\]

where \( A \) and \( B \) are determined by the PPN parameters \( \beta \) and \( \gamma \)

\[
A = 1 - \frac{2GM}{c^2r} + (\beta - \gamma) \frac{2G^2M^2}{c^4r^2} + O(c^{-6}), \tag{5.3}
\]

\[
B = 1 + 2\gamma \frac{GM}{c^2r} + O(c^{-4}). \tag{5.4}
\]

As in Schwarzschild (\( \beta = \gamma = 1 \)) we will first derive \( dr/d\tau \) and \( d\phi/d\tau \) using the constants \( e \) and \( l \) associated with the Killing vectors \( \partial_t \) and \( \partial_\phi \) and 4-velocity \( \vec{u} \) normalization

\[
-\vec{\partial}_t \cdot \vec{u} = e^2 c = Ac^2 \frac{dt}{d\tau}, \quad -\vec{\partial}_\phi \cdot \vec{u} = l = r^2 \frac{d\phi}{d\tau},
\]

(5.5)

together with 4-velocity normalization \( \vec{u}^2 = -c^2 \),

\[
-e^2 = -Ac^2 \left( \frac{dt}{d\tau} \right)^2 + B \left( \frac{dr}{d\tau} \right)^2 + r^2 \left( \frac{d\phi}{d\tau} \right)^2, \tag{5.6}
\]

where we have used \( d\theta/d\tau = 0 \). These relations give

\[
\frac{d\phi}{d\tau} = \frac{l}{r^2}, \tag{5.7}
\]

\[
\left( \frac{dr}{d\tau} \right)^{-1} = B^{1/2} \left( \frac{c^2 e^2}{A} - c^2 - \frac{l^2}{r^2} \right)^{-1/2}. \tag{5.8}
\]

Here we have written \( dr/d\tau \) as positive because, for our choice in conventions \( r_1 < r_2 \) and \( r(0) = r_1 \), this condition is demanded throughout the region of integration in Eq. (5.1).

Inserting the derivatives (5.7) and (5.8) into Eq. (5.1) gives the PPN version of (2.1)

\[
\Delta \phi = 2l \int_{r_1}^{r_2} \frac{dr}{r^2} B^{1/2} \left( c^2 e^2 A^{-1} - c^2 - \frac{l^2}{r^2} \right)^{-1/2}
\]

\[
= \left[ 1 + \gamma \frac{GM}{c^2r} + O(c^{-4}) \right] \left[ \left( c^2 e^2 \left[ 1 - \frac{2GM}{c^2r} + (\beta - \gamma) \frac{2G^2M^2}{c^4r^2} + O(c^{-6}) \right] \right)^{-1} - c^2 - \frac{l^2}{r^2} \right]^{-1/2}. \tag{5.9}
\]

We can now proceed to generalize each of the steps in II. First expand \( A^{-1} \) to order \( c^{-4} \)

\[
A^{-1} = 1 + \frac{2GM}{c^2r} + \kappa \frac{G^2M^2}{c^4r^2} + O(c^{-6}), \tag{5.10}
\]

where \( \kappa = 4 + 2(\gamma - \beta) \). Then define the Newtonian energy parameter \( E \) as in Eq. (2.2) so that

\[
c^2 e^2 A^{-1} = c^2 + \frac{2GM}{r} + \kappa \frac{G^2M^2}{c^2r^2} + \frac{2E}{m} + \frac{AEGM}{m^2c^2} + \frac{E^2}{m^2c^2} + O(c^{-4}). \tag{5.11}
\]
Substitution into Eq. (5.9), and changing variables to \( u = 1/r \), gives

\[
\Delta \phi = 2l \int_{u_2}^{u_1} du \left[ 1 + \gamma \frac{GM}{c^2} u + O(c^{-4}) \right] \left[ -\hat{A}u^2 + \hat{B}u + \hat{C} + O(c^{-4}) \right]^{-1/2},
\]

(5.12)

where the hatted constants are

\[
\hat{A} = l^2 - \kappa \left( \frac{GM}{c} \right)^2, \quad \hat{B} = B, \quad \hat{C} = C,
\]

(5.13)

where \( B \) and \( C \) are given in (2.7). By factoring \( A \) out of the denominator, and expanding, we find

\[
\Delta \phi = 2l \int_{u_2}^{u_1} du \left[ 1 + \frac{\kappa}{2} \left( \frac{GM}{lc} \right)^2 + \gamma \frac{GM}{c^2} u + O(c^{-4}) \right] \left[ -u^2 + \frac{\hat{B}}{\hat{A}} u + \frac{\hat{C}}{\hat{A}} + O(c^{-4}) \right]^{-1/2}.
\]

(5.14)

The denominator can be rewritten to emphasize the presence of the turning points

\[
\Delta \phi = 2l \int_{u_2}^{u_1} du \left[ 1 + \frac{\kappa}{2} \left( \frac{GM}{lc} \right)^2 + \gamma \frac{GM}{c^2} u + O(c^{-4}) \right] \left[ (u_1 - u)(u - u_2) + O(c^{-4}) \right]^{-1/2},
\]

(5.15)

where to leading order the ratios

\[
u_1 + u_2 = \frac{\hat{B}}{\hat{A}} = \frac{2GM}{l^2} + O(c^{-2}), \quad u_1u_2 = -\frac{\hat{C}}{\hat{A}} = -\frac{2E}{ml^2} + O(c^{-2}),
\]

(5.16)

are the same as in II. Using the integrals (2.13) and (2.16) computed in II, we now have

\[
\Delta \phi = 2\pi \left[ 1 + \frac{\kappa}{2} \left( \frac{GM}{lc} \right)^2 \right] + 2\pi \frac{GM}{l^2} \left( \frac{GM}{lc} \right)^2 = 2\pi + 2\pi(2 + 2\gamma - \beta) \left( \frac{GM}{lc} \right)^2,
\]

(5.17)

Or if we define \( \delta \phi = \Delta \phi - 2\pi \), we find that the PPN version of the result (2.17) in II is

\[
\delta \phi = 2\pi(2 + 2\gamma - \beta) \left( \frac{GM}{lc} \right)^2.
\]

(5.18)

Lastly if we use the Newtonian relation [Eq. (9.56) in Hartle] \( l^2 \simeq GMa(1 - \epsilon^2) \), where \( a \) and \( \epsilon \) are the semi-major axis and eccentricity, we have

\[
\delta \phi = \frac{1}{3}(2 + 2\gamma - \beta) \frac{6\pi GM}{c^2a(1 - \epsilon^2)}.
\]

(5.19)