Physics 318: Problem Set 5
Due Wednesday, Feb 27, 2008

1. Suppose that two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ are related by

$$\mathcal{L}'(q, \dot{q}, t) = \mathcal{L}(q, \dot{q}, t) + \frac{d}{dt}f(q, t)$$

(1)

where $f$ is some function and $q$ is shorthand for $(q_1, \ldots, q_f)$.

a. By computing the Euler-Lagrange equations for the Lagrangian $\mathcal{L}'$, show that the two Lagrangians give the same equations of motion.

b. Consider the case of a particle of charge $q$ and mass $m$ moving in electric and magnetic fields parameterized by a scalar potential $\Phi$ and a vector potential $A$. The Lagrangian for this system is

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - q\Phi(x, t) + q\dot{x} \cdot A(x, t).$$

An electromagnetic gauge transformation is a transformation of the form $A \rightarrow A + \nabla \psi$, $\Phi \rightarrow \Phi - \dot{\psi}$, where $\psi = \psi(x, t)$ is an arbitrary function. Show that under such a transformation the Lagrangian transforms as in Eq. (1), and compute the form of the function $f$.

2. Consider the Lagrangian

$$\mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2}m(q_1^2 + q_2^2) - \frac{1}{2}m\omega^2(q_1^2 + q_2^2),$$

which describes two uncoupled harmonic oscillators of mass $m$ and frequency $\omega$. Show that this system is invariant under the symmetry operation

$$q(t) \rightarrow e^{i\alpha}q(t),$$

where $q(t) \equiv q_1(t) + iq_2(t)$ and $\alpha$ is an arbitrary number. Compute the corresponding conserved quantity. How does this quantity relate to the amplitudes and phases of the individual harmonic motions of $q_1$ and $q_2$?

3. The general Galilean transformation from an inertial frame $(t, x)$ to an inertial frame $(t', x')$ can be written as

$$x'_i = \sum_{j=1}^{3} \alpha_{ij}x_j - v_it - d_i, \quad t' = t - t_0.$$

It is characterized by ten parameters: 3 rotation angles determining the orthogonal matrix $\alpha$, 3 components of the relative velocity vector $v$, 3 components of the displacement vector $d$, and a time displacement $t_0$.

a. Consider a system of $N$ particles with masses $m_n$ and positions $r_n(t)$ in the original frame. Show that in the case of no rotation ($\alpha = 1$), the total momentum $P'$ in the new frame is related to the total momentum $P$ in the original frame by

$$P' = P - Mv,$$

where $M = \sum_n m_n$ is the total mass of the system. Show that the kinetic energy $T'$ in the new frame is given in terms of the kinetic energy $T$ in the original frame by

$$T' = T - P \cdot v + \frac{1}{2}Mv^2.$$
b. Find the parameters $\alpha_{ij}$, $\bar{v}_i$, $\bar{d}_i$ and $\bar{t}_0$ of the inverse Galilean transformation

$$x_i = \sum_{j=1}^{3} \alpha_{ij}x'_j - \bar{v}_it' - \bar{d}_i, \quad t = t' - \bar{t}_0.$$ 

c. Show that the action of two consecutive Galilean transformations can be obtained from a single Galilean transformation. Express the parameters of the combined transformation in terms of those of the two transformations. Does the order of the transformations matter?

4. Derivation of Galilean Transformations: In this problem we derive the equations describing Galilean transformations from first principles. We start from the most general relation between two coordinate systems ($t, x$) and ($t', x'$):

$$t' = t'(t, x), \quad x'_i = x'_i(t, x).$$

We assume that these functions are smooth.

a. We assume that the transformation preserves the time intervals between pairs of event. Given two different events ($t, x$) and ($t + \Delta t, x + \Delta x$) in the first reference frame, show that the difference between the time coordinates in the new reference frame is $t'(t + \Delta t, x + \Delta x) - t'(t, x)$. By equating this to $\Delta t$, and using the fact that the equation is valid for all values of $t, x, \Delta t$ and $\Delta x$, argue that (i) the function $t'(t, x)$ must be independent of $x$, and (ii) the function is given by $t'(t, x) = t - t_0$ for some constant $t_0$.

b. Now fix a value of $t$, and define the function $f_i(x) = x'_i(t, x)$. We assume that the transformation preserves distances, which implies that for every pair of points $x_1$ and $x_2$, $|f(x_1) - f(x_2)| = |x_1 - x_2|$. By squaring this equation and differentiating with respect to $x_1$ and then with respect to $x_2$ show that

$$\sum_{i=1}^{3} \frac{\partial f_i(x_1)}{\partial x_j} \frac{\partial f_i(x_2)}{\partial x_k} = \delta_{jk},$$

where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise. Multiply across by the inverse matrix of the matrix $\frac{\partial f_i}{\partial x_j}(x_1)$, and then argue that since the left hand side is independent of $x_1$ and the right hand side is independent of $x_2$, both sides must be independent of both $x_1$ and $x_2$. Deduce that

$$x'_i(t, x) = \sum_{j=1}^{3} \alpha_{ij}(t)x_j - d_i(t), \quad (2)$$

where $d_i(t)$ is an arbitrary function of time and the matrix $\alpha_{ij}$ is orthogonal.

c. Next we impose the fact that the transformation should preserve the form of Newton’s first law. Suppose that $x_i = x_i(t)$ is the path of a particle in the original frame, with velocity $v_i(t) = dx_i/dt$ and acceleration $a_i(t) = dv_i/dt$. Show using Eq. (2) that the acceleration as measured in the new frame is

$$\frac{d^2x'_i}{dt'^2} = \sum_{j=1}^{3} [\alpha_{ij}x_j + 2\dot{\alpha}_{ij}v_j + \alpha_{ij}a_j] - \ddot{d}_i.$$ 

Deduce that in order to preserve Newton’s first law we must have

$$\sum_{j=1}^{3} [\alpha_{ij}x_j + 2\dot{\alpha}_{ij}v_j] - \ddot{d}_i = 0.$$ 

Argue that since this equation must hold for all values of $x$ and $v$, that $\dot{\alpha}_{ij} = 0$ and $\ddot{d}_i = 0$. Deduce the usual formula for Galilean transformations.