Physics 318: Problem Set 6
Due Wednesday, March 5, 2008

1. Consider a particle of mass $\mu$ moving in the attractive Coulomb potential $V(\rho) = -\alpha/\rho$ where $\alpha$ is a positive constant.

a. Show that when the energy $E$ of the particle is negative (corresponding to bound elliptical orbits), the relation between time $t$ and radius $\rho$ is given by

$$t = \sqrt{\frac{\mu}{2|E|}} \int d\rho \frac{\rho}{\sqrt{a^2\varepsilon^2 - (\rho - a)^2}}.$$ 

Here

$$\varepsilon = \sqrt{1 + \frac{2El^2}{\mu\alpha^2}}, \quad a = \frac{\alpha}{2|E|}$$

are the eccentricity and semi-major axis of the ellipse, and $l$ is the angular momentum of the orbit.

b. By making the change of variables $\rho = a - a\varepsilon \cos \xi$ in the integral, deduce that the relation between $t$ and $\rho$ can be written in the parametric form

$$t(\xi) = \sqrt{\frac{\mu a^3}{\alpha}} (\xi - \varepsilon \sin \xi), \quad \rho(\xi) = a(1 - \varepsilon \cos \xi).$$

These are called Kepler’s equations.

2. For motion in a central potential $V(\rho)$, show that the change $\Delta \varphi$ in the angle $\varphi$ while $\rho$ changes from its maximum value $\rho_{\text{max}}$ to its minimum value $\rho_{\text{min}}$ and back again is

$$\Delta \varphi = -2 \frac{\partial}{\partial \rho} \int_{\rho_{\text{min}}}^{\rho_{\text{max}}} d\rho \left\{ 2\mu [E - V(\rho)] - \frac{l^2}{\rho^5} \right\}^{1/2}.$$

(1)

b. Suppose that the potential is a Coulomb potential plus a small inverse cubic term:

$$V(\rho) = -\frac{\alpha}{\rho} + \frac{\beta}{\rho^3}.$$ 

By substituting into Eq. (1), expanding to linear order in $\beta$, and changing the variable of integration from $\rho$ to $\varphi$ (using the relation $\rho = \rho(\varphi)$ for $\beta = 0$), show that $\Delta \varphi = 2\pi - 6\pi\beta/(\alpha p^2) + O(\beta^2)$, where $p = l^2/(\mu\alpha)$ is the semi-latus rectum of the orbit.

c. The leading order correction to Newtonian gravity from general relativity is of the above form, where $\alpha = Gm_3m_1$ and $\beta = -GMl^2/(\mu c^2)$, where $G$ is Newton’s constant of gravitation, $c$ is the speed of light, $l$ is the orbital angular momentum, and $M = m_1 + m_2$ and $\mu = m_1m_2/M$ are the total and reduced masses. Evaluate the perihelion shift for Mercury’s orbit around the Sun. The Sun’s mass is $2.00 \times 10^{30}$ kg, Mercury’s mass is $3.30 \times 10^{23}$ kg, the semi-major axis of the orbit is $5.79 \times 10^{10}$ m, and the eccentricity is 0.206. Express your answer in arc seconds per century. [The agreement between the prediction of general relativity and the observed perihelion precession is one of the three classical tests of general relativity.]
3. 
   a. Show that if a particle moves in a circular orbit under the influence of an attractive central force directed toward a point such a circular orbit, then the force varies as the inverse-fifth power of from that point. The diagram explains the geometry of such an orbit.

   ![Diagram of a circular orbit with angles labeled](image)

   b. Show that for the orbit described, the total energy of the particle is zero.

   c. Find the period of the motion.

   d. If the circle is in the $xy$ plane, find $\dot{x}$, $\dot{y}$ and $v$ as a function of the angle around the circle, and show that all three quantities become infinite as the particle passes through the center of force.

4. [Problem due to Prof. Neubert] Consider the elastic scattering of a beam of particles with initial velocities $v_\infty e_x$ on a rigid surface, whose shape is obtained by rotating the curve $y(x) = \sqrt{a x}$, for $x \geq 0$, about the $x$-axis. Here $a$ is a positive constant. Assume that the particles bounce elastically off the surface.

   ![Diagram of a semi-circle](image)

   Use a geometrical argument to find a relation between the impact parameter $s$ (the distance from the $x$-axis) and the scattering angle $\theta$. Derive the differential scattering cross section $d\sigma/d\Omega$, and show that it has the same angular dependence as Rutherford scattering.